

ON THE RELATIONSHIP BETWEEN THE NUMBER OF SOLUTIONS OF CONGRUENCE SYSTEMS AND THE RESULTANT OF TWO POLYNOMIALS

Dmitry Khomovsky

M.V.Lomonosov Moscow State University, Moscow, RF
khomovskij@physics.msu.ru

Abstract

Let q be an odd prime and $f(x)$, $g(x)$ be polynomials, with integer coefficients. If the system of congruences $f(x) \equiv g(x) \equiv 0 \pmod{q}$ has ℓ solutions, then $R(f(x), g(x)) \equiv 0 \pmod{q^\ell}$, where $R(f(x), g(x))$ is the resultant of the polynomials. Using this result we give a new proofs of some known congruences with the Lucas sequences.

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1. Introduction

The resultant [10] $R(f, g)$ of two polynomials $f(x) = a_n x^n + \dots + a_0$ and $g(x) = b_m x^m + \dots + b_0$ of degrees n and m , respectively, with coefficients in a field F is defined by the determinant of the $(m+n) \times (m+n)$ Sylvester matrix

$$R(f, g) = \begin{vmatrix} a_n & a_{n-1} & \dots & \dots & \dots & a_0 & & & & \\ & a_n & a_{n-1} & \dots & \dots & \dots & a_0 & & & \\ & & \dots & & & & & & & \\ & & & a_n & a_{n-1} & \dots & \dots & \dots & a_0 & \\ b_m & b_{m-1} & \dots & \dots & b_0 & & & & & \\ & b_m & b_{m-1} & \dots & \dots & b_0 & & & & \\ & & \dots & & & & & & & \\ & & & \dots & & & & & & \\ & & & & b_m & b_{m-1} & \dots & \dots & b_0 & \end{vmatrix} \quad (1)$$

Let f, g, h and v be polynomials below. Some important properties of resultant:

(i) If $f(x) = a_n \prod_{i=1}^n (x - \alpha_i)$ and $g(x) = b_m \prod_{j=1}^m (x - \beta_j)$, then

$$R(f, g) = a_n^m \prod_{i=1}^n g(\alpha_i) = (-1)^{mn} b_m^n \prod_{i=1}^m f(\beta_i) = a_n^m b_m^n \prod_{i=1}^n \prod_{j=1}^m (\alpha_i - \beta_j),$$

where α_i and β_j are the roots of $f(x)$ and $g(x)$, respectively, in some extension of F , each repeated according to its multiplicity. These property is taken often as the definition of

resultant.

(ii) f and g have a common root in some extension of F if and only if $R(f, g) = 0$.

(iii) $R(f, g) = (-1)^{nm} R(g, f)$.

(iv) $R(fh, g) = R(f, g) R(h, g)$ and $R(f, gh) = R(f, g) R(f, h)$.

(v) If $g = vf + h$ and $\deg(h) = d$, then $R(f, g) = a_n^{m-d} R(f, h)$.

(vi) If p is positive integer, then $R(f(x^p), g(x^p)) = R(f(x), g(x))^p$.

All these properties are well known [1, 7]. More details concerning resultant can be found in [3, 4]. Another important classical result:

Lemma 1. Let $f = \sum_{i=0}^n a_i x^i$ and $g = \sum_{j=0}^m b_j x^j$ be two polynomials of degrees n and m respectively. Let, for $k \geq 0$, $r_k(x) = r_{k,n-1}x^{n-1} + \dots + r_{k,0}$ be the remainder of $x^k g(x)$ modulo $f(x)$, i.e., $x^k g(x) = v_k(x)f(x) + r_k(x)$, where v_k is some polynomial and $\deg(r_k) \leq n-1$. Then

$$R(f, g) = a_n^m \begin{vmatrix} r_{n-1,n-1} & r_{n-1,n-2} & \cdots & r_{n-1,0} \\ r_{n-2,n-1} & r_{n-2,n-2} & \cdots & r_{n-2,0} \\ \vdots & & & \vdots \\ r_{0,n-1} & r_{0,n-2} & \cdots & r_{0,0} \end{vmatrix} \quad (2)$$

Proof. See [4]. □

In next section we proof a theorem on the relationship between the number of solutions of congruence system $f(x) \equiv g(x) \equiv 0 \pmod{q}$ and the resultant of two polynomials $R(f(x), g(x))$. Then using this result we give a new proof of some congruences with the Lucas sequences.

2. Properties of the resultant

Let q be an odd prime. A polynomial $f(x)$ with integer coefficients is called non-zero in \mathbb{Z}_q , if at least one of coefficients of $f(x)$ is not divisible by q . Let $A = (a_{i,j})$ be an arbitrary matrix. Then by $A^{<q>}$ we will denote the matrix $(a'_{i,j})$ over \mathbb{Z}_q of the same type such that $a'_{i,j}$ is the residue of $a_{i,j}$ modulo q .

Theorem 1. Let $f(x)$ and $g(x)$ be two polynomials with integer coefficients and these polynomials be non-zero in \mathbb{Z}_q . If the system of congruences $f(x) \equiv 0 \pmod{q}$ and $g(x) \equiv 0 \pmod{q}$ has ℓ solutions then $R(f(x), g(x)) \equiv 0 \pmod{q^\ell}$.

Proof. Let $\deg f = n$, $\deg g = m$, then we have that the system $f(x) \equiv g(x) \equiv 0 \pmod{q}$ has ℓ solutions by the theorem conditions and $\ell \leq \min[n, m]$, as the polynomials are non-zero in \mathbb{Z}_q . Let $r_k(x) = r_{k,n-1}x^{n-1} + \dots + r_{k,0}$ be the remainder of $x^k g(x)$ modulo $f(x)$,

i.e., $x^k g(x) = v_k(x)f(x) + r_k(x)$, where $v_k(x)$ is some polynomial and $\deg(r_k) \leq n-1$. Then we get the system of congruences

$$\begin{pmatrix} r_{n-1,n-1} & r_{n-1,n-2} & \cdots & r_{n-1,0} \\ r_{n-2,n-1} & r_{n-2,n-2} & \cdots & r_{n-2,0} \\ \vdots & & & \vdots \\ r_{0,n-1} & r_{0,n-2} & \cdots & r_{0,0} \end{pmatrix} \begin{pmatrix} x^{n-1} \\ x^{n-2} \\ \vdots \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{q} \quad (3)$$

This system of congruences has not less than ℓ solutions, since each congruence of (3) is derived from $f(x) \equiv 0 \pmod{q}$ and $g(x) \equiv 0 \pmod{q}$. Let $A = (a_{i,j})$ be a matrix of the system (3). Using the procedure analogical to row reduction, by operations of swapping the rows and adding a multiple of one row to another row, we can reduce A to a matrix A_1 with integer coefficients such that $\det(A) = \pm \det(A_1)$ and $A_1^{<q>}$ is an upper triangular matrix. We can note that each solution of the system (3) is also a solution of the following system over \mathbb{Z}_q :

$$(A_1^{<q>}) \begin{pmatrix} x^{n-1} \\ x^{n-2} \\ \vdots \\ 1 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \pmod{q}, \quad (4)$$

so the system (4) has at least ℓ solutions. Note that last ℓ congruences of system (4) have the degrees less than ℓ . On the other hand, these congruences have at least ℓ solutions. Hence all these congruences have to be congruences with zero coefficients, i.e. the last ℓ rows of $A_1^{<q>}$ are zero rows. Therefore, all elements of last ℓ rows of A_1 are divisible by q , so $\det(A) = \pm \det(A_1)$ is divisible by q^ℓ . Thus, by Lemma 1 we have $R(f, g) \equiv 0 \pmod{q^\ell}$. \square

Remark: If one or both polynomials equal to zero in \mathbb{Z}_q , then by property (i) we obtain that or $R(f, g) \equiv 0 \pmod{q^n}$, or $R(f, g) \equiv 0 \pmod{q^m}$. This trivial case we don't consider in Theorem 1.

Example: $f(x) = x^6 + 1$, $g(x) = (x+1)^6 + 1$. The system of congruences $x^6 + 1 \equiv 0 \pmod{13}$ and $(x+1)^6 + 1 \equiv 0 \pmod{13}$ has 3 solution in \mathbb{Z}_{13} : $x = 5, 6, 7$. The matrix of the system (3) for these polynomials:

$$A = \begin{pmatrix} 1 & -6 & -15 & -20 & -15 & -6 \\ 6 & 1 & -6 & -15 & -20 & -15 \\ 15 & 6 & 1 & -6 & -15 & -20 \\ 20 & 15 & 6 & 1 & -6 & -15 \\ 15 & 20 & 15 & 6 & 1 & -6 \\ 6 & 15 & 20 & 15 & 6 & 1 \end{pmatrix} \quad (5)$$

Since the resulting echelon form after row reduction is not unique, we obtain also reduced row echelon form which is unique.

$$A_1^{<13>} = \begin{pmatrix} 1 & 7 & 11 & 6 & 11 & 7 \\ 0 & 1 & 9 & 11 & 1 & 11 \\ 0 & 0 & 1 & 8 & 3 & 11 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 7 & 4 & 8 \\ 0 & 1 & 0 & 4 & 0 & 3 \\ 0 & 0 & 1 & 8 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (6)$$

So we get $\det A \equiv 0 \pmod{13^3}$ and $R(x^6 + 1, (x+1)^6 + 1) \equiv 0 \pmod{13^3}$. This resultant is actually equal to $2^4 \times 5 \times 13^3$.

Corollary 1. *Let $f(x)$, $g(x)$ be two polynomials of degrees n and m , respectively, with integer coefficients and these polynomials be non-zero in \mathbb{Z}_q . Let A be a matrix of the system (3) for polynomials $f(x), g(x)$. If $\text{Rank } A = p$ in \mathbb{Z}_q , then $R(f, g) \equiv 0 \pmod{q^{n-p}}$. On the other hand if the system $f(x) \equiv g(x) \equiv 0 \pmod{q}$ has ℓ solutions then $n - p \geq \ell$. Moreover, if M is any $k \times k$ minor of the matrix A and $k > p$, then $M \equiv 0 \pmod{q^{k-p}}$.*

Proof. This follows from Theorem 1. \square

The question about the relation of the multiplicity of q as a factor of $R(f, g)$ and the degree of common factor of polynomials f and g modulo q was studied in [2]. This question closely related to Theorem 1 and first appeared in [5].

3. The congruences with the members of the Lucas sequences

Theorem 2. *Let $f(x) = a_n x^n + \dots + a_0$ be a polynomial of degree n with integer coefficients and q be an odd prime. Let $a_0 \not\equiv 0 \pmod{q}$ and the congruence $f(x) \equiv 0 \pmod{q}$ has ℓ solutions. Then*

$$R(f(x), x^{q-1} - 1) \equiv a_n^{q-1} \prod_{i=1}^n (\alpha_i^{q-1} - 1) \equiv 0 \pmod{q^\ell}, \quad (7)$$

where α_i are the roots of $f(x)$ each repeated according to its multiplicity.

Proof. Consider $R(f(x), x^{q-1} - 1)$. Since $f(x) = a_n \prod_{i=1}^n (x - \alpha_i)$, so

$$R(f(x), x^{q-1} - 1) = a_n^{q-1} \prod_{i=1}^n (\alpha_i^{q-1} - 1). \quad (8)$$

We know q is an odd prime, so the congruence $x^{q-1} - 1 \equiv 0 \pmod{q}$ has $q - 1$ solutions except 0. On the other hand the congruence $f(x) \equiv 0 \pmod{q}$ has ℓ solutions not equal to 0, as $a_0 \not\equiv 0 \pmod{q}$. Hence the system of congruences $f(x) \equiv x^{q-1} - 1 \equiv 0 \pmod{q}$ has also ℓ solutions, then by Theorem 1 we have $R(f(x), x^{q-1} - 1) \equiv 0 \pmod{q^\ell}$. \square

Theorem 3. *Let $f(x) = a_n x^n + \dots + a_0$ be a polynomial of degree n with integer coefficients and q be an odd prime. Let $a_0 \not\equiv 0 \pmod{q}$ and the congruence $f(x) \equiv 0 \pmod{q}$ has ℓ solutions. If b solutions are quadratic residues modulo q and, correspondingly, $\ell - b$ solutions are quadratic nonresidues modulo q , then*

$$R(f(x), x^{\frac{q-1}{2}} - 1) \equiv a_n^{\frac{q-1}{2}} \prod_{i=1}^n (\alpha_i^{\frac{q-1}{2}} - 1) \equiv 0 \pmod{q^b} \quad (9)$$

and

$$R(f(x), x^{\frac{q-1}{2}} + 1) \equiv a_n^{\frac{q-1}{2}} \prod_{i=1}^n (\alpha_i^{\frac{q-1}{2}} + 1) \equiv 0 \pmod{q^{\ell-b}}, \quad (10)$$

where α_i are the roots of the $f(x)$ each repeated according to its multiplicity.

Proof. Consider $R(f(x), x^{\frac{q-1}{2}} - 1)$. Since $f(x) = a_n \prod_{i=1}^n (x - \alpha_i)$, so

$$R\left(f(x), x^{\frac{q-1}{2}} - 1\right) = a_n^{\frac{q-1}{2}} \prod_{i=1}^n (\alpha_i^{\frac{q-1}{2}} - 1). \quad (11)$$

As q is an odd prime and $f(x) \equiv 0 \pmod{q}$ has b solutions, which are quadratic residues modulo q , the system of congruences $f(x) \equiv x^{\frac{q-1}{2}} - 1 \equiv 0 \pmod{q}$ has b solutions, so by Theorem 1 we have $R(f(x), x^{\frac{q-1}{2}} - 1) \equiv 0 \pmod{q^b}$.

By analogy we prove that $R(f(x), x^{\frac{q-1}{2}} + 1) \equiv 0 \pmod{q^{\ell-b}}$ \square

As an illustration of applications for Theorem 2 we consider the next theorem.

Theorem 4. *Let q be an odd prime and Q, P be any integers such that $Q \not\equiv 0 \pmod{q}$. If the Legendre symbol $\left(\frac{P^2-4Q}{q}\right)$ is equal to 1, then*

$$V_{q-1}(P, Q) \equiv Q^{q-1} + 1 \pmod{q^2}, \quad (12)$$

$$V_{\frac{q-1}{2}}^2(P, Q) \equiv \left(Q^{\frac{q-1}{2}} + 1\right)^2 \pmod{q^2}, \quad (13)$$

where $V_n(P, Q)$ is the n -th term of the Lucas sequence defined by recurrence relation

$$V_0 = 2, \quad V_1 = P, \quad V_i = PV_{i-1} - QV_{i-2}, \quad i \geq 2. \quad (14)$$

Proof. The roots of $x^2 - Px + Q$ are $\alpha_1 = \frac{P - \sqrt{P^2 - 4Q}}{2}$, $\alpha_2 = \frac{P + \sqrt{P^2 - 4Q}}{2}$. Hence $R(x^2 - Px + Q, x^{q-1} - 1) = (\alpha_1 \alpha_2)^{q-1} - (\alpha_1^{q-1} + \alpha_2^{q-1}) + 1 = 1 + Q^{q-1} - V_{q-1}(P, Q)$. Since $\left(\frac{P^2-4Q}{q}\right) = 1$ and $Q \not\equiv 0 \pmod{q}$, the system of congruences $x^2 - Px + Q \equiv x^{q-1} - 1 \equiv 0 \pmod{q}$ has two solutions, hence by Theorem 1 we have $1 + Q^{q-1} - V_{q-1}(P, Q) \equiv 0 \pmod{q^2}$, so we get (12). Now using well known identity $V_{2n}(P, Q) = V_n^2(P, Q) - 2Q^n$ we get (13). \square

Note that the congruences (12) and (13) are already known [6, 8, 9], but here we give an alternative completely independent proof of these results.

Theorem 4 is the particular case of more general.

Theorem 5. *Let q be an odd prime and k, P, Q be any integers such that $k^2 + Pk + Q \not\equiv 0 \pmod{q}$ and $Q \not\equiv 0 \pmod{q}$. If $\left(\frac{P^2-4Q}{q}\right) = 1$, then*

$$V_{q-1}(P + 2k, Q + Pk + k^2) \equiv (k^2 + Pk + Q)^{q-1} + 1 \pmod{q^2}, \quad (15)$$

$$V_{\frac{q-1}{2}}^2(P + 2k, Q + Pk + k^2) \equiv (k^2 + Pk + Q)^{q-1} + 2(k^2 + Pk + Q)^{\frac{q-1}{2}} + 1 \pmod{q^2}. \quad (16)$$

Proof. Consider the resultant

$R(x^2 - Px + Q, (x + k)^{q-1} - 1) =$
 $= (\alpha_1 \alpha_2 + k(\alpha_1 + \alpha_2) + k^2)^{q-1} - ((\alpha_1 + k)^{q-1} + (\alpha_2 + k)^{q-1}) + 1 =$
 $= 1 + (k^2 + Pk + Q)^{q-1} - V_{q-1}(P + 2k, Q + Pk + k^2)$. As $k^2 + Pk + Q \not\equiv 0 \pmod{q}$. Since the value $-k$ is not a solution of $x^2 - Px + Q \equiv 0 \pmod{q}$ and $\left(\frac{P^2-4Q}{q}\right) = 1$, so the system of congruences $x^2 - Px + Q \equiv (x + k)^{q-1} - 1 \equiv 0 \pmod{q}$ has two solutions, hence by Theorem 1 we have $R(x^2 - Px + Q, (x + k)^{q-1} - 1) \equiv 0 \pmod{q^2}$, so we get (15). Now using identity $V_{2n}(P, Q) = V_n^2(P, Q) - 2Q^n$, we get (16). \square

Theorem 5 allows to obtain the following corollaries.

The congruences with the Lucas numbers.

Let $P = 1$, $Q = -1$ and $\left(\frac{5}{q}\right) = 1$, i.e. by the law of quadratic reciprocity $q \equiv \pm 1 \pmod{5}$. Let further an integer k satisfies $k^2 + k - 1 \not\equiv 0 \pmod{q}$. Then

$$V_{q-1}(1 + 2k, k^2 + k - 1) \equiv (k^2 + k - 1)^{q-1} + 1 \pmod{q^2}, \quad (17)$$

$$V_{\frac{q-1}{2}}^2(1 + 2k, k^2 + k - 1) \equiv (k^2 + k - 1)^{q-1} + 2(k^2 + k - 1)^{\frac{q-1}{2}} + 1 \pmod{q^2}. \quad (18)$$

If $k = 0$, then

$$L_{q-1} \equiv 2 \pmod{q^2}, \quad (19)$$

$$L_{\frac{q-1}{2}}^2 \equiv 2 + 2(-1)^{\frac{q-1}{2}} \pmod{q^2}, \quad (20)$$

where L_n is the n -th Lucas number.

The congruences with the Pell-Lucas numbers.

Let $P = 2$, $Q = -1$ and $\left(\frac{8}{q}\right) = 1$, i.e. by the law of quadratic reciprocity $q \equiv \pm 1 \pmod{8}$. Let further an integer k satisfies $k^2 + 2k - 1 \not\equiv 0 \pmod{q}$, then

$$V_{q-1}(2 + 2k, k^2 + 2k - 1) \equiv (k^2 + 2k - 1)^{q-1} + 1 \pmod{q^2}, \quad (21)$$

$$V_{\frac{q-1}{2}}^2(2 + 2k, k^2 + 2k - 1) \equiv (k^2 + 2k - 1)^{q-1} + 2(k^2 + 2k - 1)^{\frac{q-1}{2}} + 1 \pmod{q^2}. \quad (22)$$

If $k = 0$, then

$$\tilde{P}_{q-1} \equiv 2 \pmod{q^2}, \quad (23)$$

$$\tilde{P}_{\frac{q-1}{2}}^2 \equiv 2 + 2(-1)^{\frac{q-1}{2}} \pmod{q^2}, \quad (24)$$

where \tilde{P}_n is the n -th Pell-Lucas number defined by:

$$\tilde{P}_0 = 2, \quad \tilde{P}_1 = 2, \quad \tilde{P}_i = 2\tilde{P}_{i-1} + \tilde{P}_{i-2}, \quad i \geq 2. \quad (25)$$

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References

- [1] P.M. Cohn, *Algebra, Vol. 1*, Wiley, New York, 1980.
- [2] D. Gómez-Pérez, J. Gutierrez, A. Ibeas and D. Sevilla. *Common factors of resultants modulo p* , Bull. Aust. Math. Soc. **79** (2009), 299-302.
- [3] C. Helou, G. Terjanian, *Arithmetical properties of wendt's determinant*, Journal of Number Theory **115** (2005), 45-57.
- [4] S. Janson, *Resultant and discriminant of polynomials*, (2007).
- [5] S. V. Konyagin and I. Shparlinski. *Character Sums with Exponential Functions and their Applications*, Cambridge University Press (1999).
- [6] R. J. McIntosh, E. L. Roettger, *A search for Fibonacci-Wieferich and Wolstenholme primes*, Math. Comp. **76** (2007), 2087-2094.

- [7] P. Ribenboim, *Fermat's Last Theorem for Amateurs*, Springer, New York, 1999.
- [8] Z.-H. Sun, Z.-W. Sun, *Fibonacci numbers and Fermats last theorem*, Acta Arith **60** (1992), 371-388.
- [9] Z.W. Sun, R. Tauraso, *New congruences for central binomial coefficients*, Adv. in Appl. Math.
- [10] B.L. van der Waerden, *Algebra, vol. 1*, F. Ungar Pub. Co., New York, 1977.